Acta Crystallographica Section A
Foundations of Crystallography
ISSN 0108-7673

Received 14 June 2006
Accepted 28 December 2006

Hybrid quasiperiodic-periodic structures constructed by projection in two stages

S. I. Ben-Abraham[a] and Alexander Quandt[b]*

[a]Department of Physics, Ben-Gurion University, POB 653, IL-84105 Beer-Sheba, Israel, and [b]Institut für Physik, Ernst-Moritz-Arndt Universität, Domstrasse 10a, D-17489 Greifswald, Germany. Correspondence e-mail: alexander.quandt@physik.uni-greifswald.de

A two-stage variant of the cut-and-project method is presented, in which a periodic structure is cut and projected in a high-dimensional space onto three-dimensional physical space so that a second cut and projection onto a plane yields a quasiperiodic structure. The method is applied to the cases of octagonal, dodecagonal and pentagonal/decagonal symmetry. The focus is on the three-dimensional intermediate hybrid structures that are partly quasiperiodic and partly periodic. The method can be generalized to other symmetries as well as to include more intermediate steps.

1. Introduction

The term 'quasicrystal' was originally coined to denote in general a quasiperiodic crystalline structure as distinct from a periodic one. Yet the term is mostly restricted to mean an alloy with a 'non-crystallographic' symmetry, specifically octagonal, decagonal (including pentagonal), dodecagonal or icosahedral. Only the latter is genuinely quasiperiodic in all three dimensions of physical space. The others are, in fact, considered to be two-dimensional, the third dimension being treated separately. Such an interpretation, while satisfactory for most practical purposes, is, nevertheless, misleading for a number of reasons. In the first place, a mathematical quasicrystal, that is a quasiperiodic structure, might display any kind of point symmetry, unless restricted, say, by a 'quasicrystallographic lemma' to be associated only with quadratic irrationals. Secondly, crystallographic point symmetry does not necessarily imply periodicity. Rather, it turns out that, in the generic case, quasiperiodic structures display crystallographic symmetries while the non-crystallographic ones, although being the most interesting ones, are the rare exceptions. A third objection might perhaps often be irrelevant but is central for this study. It is quite obvious that in a real physical 'two-dimensional' quasicrystal the structure along the third dimension is intimately connected to that of the quasiperiodic planes.

An important subclass of quasiperiodic structures, if not the most important one from a practical point of view, is that of the cut-and-project (alias model) sets. Hence, it naturally comes to one's mind to consider structures gained by projection of a periodic structure in some high-dimensional space onto three-dimensional physical space such that a further projection onto a plane yields the known two-dimensional quasicrystals. This was the aim of this work. We call our procedure the 'two-stage cut-and-project method'. Previously, we have carried it out in one instance each of the dodecagonal (Ben-Abraham *et al.*, 2004) and octagonal cases (Ben-Abraham, 2004). Our new contribution concerns another, perhaps more interesting, twelvefold case and a new look at the 'traditional' fivefold case, which, of course, includes the tenfold case as well.

The rest of this paper is organized as follows. In §2, we briefly describe the canonical as well as the two-stage cut-and-project scheme. §§3 and 4 are concise reminders of the earlier octagonal and dodecagonal versions. We present our new results in §§5 through 8. In §5, we apply our method to an improved dodecagonal version, in §§6 to 8, we revisit the pentagonal/decagonal case. We conclude with some remarks on the relevance of our findings.

2. Two-stage cut-and-project scheme

For a thorough treatment of the cut-and-project scheme, we refer to Moody (1997). Somewhat more popular but still rigorous introductions can be found in the books by Janot (1994) and Senechal (1995). For our purposes, we restrict ourselves to a drastically limited version. We consider a lattice L_D in a real space of $D > 3$ dimensions $\mathbf{R}^D = \mathbf{R}^d_{||} \times \mathbf{R}^c_{\perp}$, where $\mathbf{R}^d_{||}$ is a real space of $d \leq 3$ dimensions, called the parallel, physical or direct space and $\mathbf{R}^c_{\perp}$, called the perpendicular, internal or dual space, is a real space of dimension $c = D - d$, c being the codimension of d. We let in turn $\mathbf{R}^d_{||}$ be $\mathbf{R}^d_{||} = \mathbf{R}^\delta \times \mathbf{R}^\gamma$, where $d = \delta + \gamma$; usually we shall have $d = 3$, $\delta = 2$, $\gamma = 1$. We project L_D orthogonally onto $\mathbf{R}^d_{||}$. This, however, would, in general, produce a dense set. Therefore we select a fundamental region (unit cell) in $\mathbf{R}^D$ and project it onto $\mathbf{R}^c_{\perp}$. This projection, $\boldsymbol{W}$, is called the acceptance domain or window of the scheme. Eventually, we project onto $\mathbf{R}^d_{||}$ only those points of L_D which project into $\boldsymbol{W}$ in $\mathbf{R}^c_{\perp}$. This is the canonical cut-and-project scheme. Our twist is that we repeat the procedure with $\mathbf{R}^\delta$ and $\mathbf{R}^\gamma$ playing the role of parallel and perpendicular space, respectively. Thus we arrive at the final projection in two stages. Hence, we call our method the *two-stage cut-and-project scheme*.

3. Octagonal structure

The simplest example seems to be the octagonal case (Ben-Abraham, 2004). We cut and project the four-dimensional simple cubic lattice represented as $(2\mathbf{Z}+1)^4$ onto $\mathbf{R}^3_{||}$. The window is either the half-open interval

$$\boldsymbol{W} := \left(-(1+1/2^{1/2}), +(1+1/2^{1/2}), +(1+1/2^{1/2})\right] \subset \mathbf{R}^1_{\perp} \tag{1}$$

or

$$\boldsymbol{W}' := \left[-(1+1/2^{1/2}), +(1+1/2^{1/2})\right) \subset \boldsymbol{R}^1_{\perp}. \tag{1$'$}$$

The central unit cell $\boldsymbol{C}$ is a 4-cube spanned by the vectors $\langle 1111\rangle$, *i.e.* the vector [1111] and all vectors symmetrically equivalent to it. The convex hull of its projection into $\mathbf{R}^3_{||}$ is a hexagonal prism segment $\boldsymbol{C}_3$. Its 12 outer vertices belong to two types, A and B; referred to a Cartesian orthogonal system $(Oxyz)$ they are

$$\begin{aligned} &A: \pm 1/2^{1/2}, \pm 2^{1/2}, \pm 1;\\ &B: \pm\left(1+1/2^{1/2}\right), 0, \pm 1. \end{aligned} \tag{2}$$

The two bases of $\boldsymbol{C}_3$ contain two more vertices each:

$$\begin{aligned} &C: -(1-1/2^{1/2}), 0, \pm 1;\\ &C': +(1-1/2^{1/2}), 0, \pm 1. \end{aligned} \tag{3}$$

In the resulting structure, only one of the two points C and C' is realized depending on which end of the window, $\boldsymbol{W}$ or $\boldsymbol{W}'$, is closed. The alternation between C and C' offers the possibility of flipping and hence of phasons.

The resulting structure is quasiperiodic in the x direction (in Fig. 1 along $C'B$, also identical to ξ) with quasiperiods (*i.e.* characteristic distances between adjacent points) 1 and $2^{1/2}$. It is periodic in the y direction (in Fig. 1 along AA within the basal plane) with period 2 and in the z direction (in Fig. 1 along the vertical BB) with period $2^{1/2}$. Part of the basal plane (xy) is shown in Fig. 2.

The directions η and ζ lie in the (yz) plane inclined to the z direction by $\pm$ 45°.

A second cut and orthogonal projection along either of these directions (see Fig. 1) eventually produces the well known octagonal Ammann–Beenker tiling.

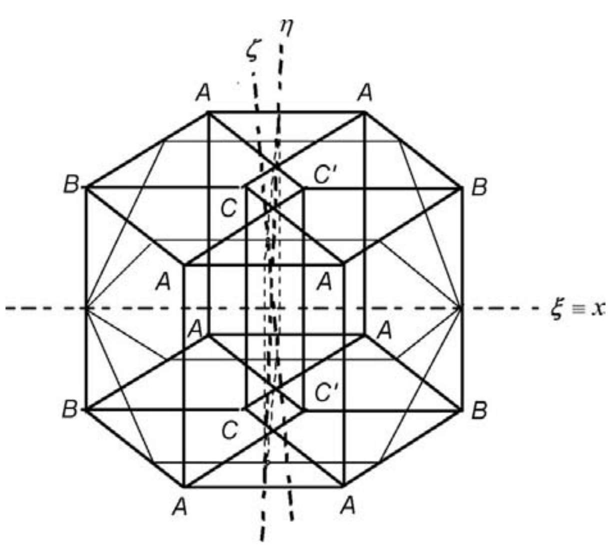

Figure 1
Cavalier perspective of the hexagonal prism segment $\boldsymbol{C}_{;3}$ – the three-dimensional projection of the central unit cell $\boldsymbol{C}$.

4. Pseudododecagonal structure

This structure has been described in detail by Ben-Abraham *et al.* (2004). We cut and project the six-dimensional simple cubic lattice represented as $(2\mathbf{Z}+1)^6$ onto $\mathbf{R}^3_{||}$ spanned on the six-dimensional orthogonal basis vectors

$$\begin{aligned} \mathbf{a}_1 &= \tfrac{1}{2}\left[2, 3^{1/2}, -1, -3^{1/2}, -1, 0\right],\\ \mathbf{a}_2 &= \tfrac{1}{2}\left[0, 1, 3^{1/2}, 1, -3^{1/2}, -2\right],\\ \mathbf{a}_3 &= [1, -1, 1, -1, 1, -1]. \end{aligned} \tag{4}$$

The perpendicular space $\mathbf{R}^3_{\perp}$ is spanned on the six-dimensional orthogonal basis vectors

$$\begin{aligned} \mathbf{a}_4 &= \tfrac{1}{2}\left[-3^{1/2}, 2, 0, -1, 3^{1/2}, -1\right],\\ \mathbf{a}_5 &= \tfrac{1}{2}\left[1, 0, -2, 3^{1/2}, 1, -3^{1/2}\right],\\ \mathbf{a}_6 &= [1, 1, 1, 1, 1, 1]. \end{aligned} \tag{5}$$

The corresponding normalized unit vectors are

$$\begin{aligned} \mathbf{e}_k &= \frac{\mathbf{a}_k}{3^{1/2}}, \quad k = 1, 2, 4, 5;\\ \mathbf{e}_k &= \frac{\mathbf{a}_k}{6^{1/2}}, \quad k = 3, 6. \end{aligned} \tag{6}$$

The central unit cell $\boldsymbol{C}$ is a 6-cube spanned by the vectors $\langle 111111\rangle$. The convex hull of its projection into $\mathbf{R}^3_{||}$ is a triacontahedron $\boldsymbol{T}_{||}$. Its two-dimensional facets are squares and two kinds of rhombi; its symmetry is $\bar{3}m$. The window $\mathbf{W}$ is, of course, the projection of $\boldsymbol{C}$ into $\mathbf{R}^3_{\perp}$. Its convex hull is again a triacontahedron $\boldsymbol{T}_{\perp}$ congruent to $\boldsymbol{T}_{||}$.

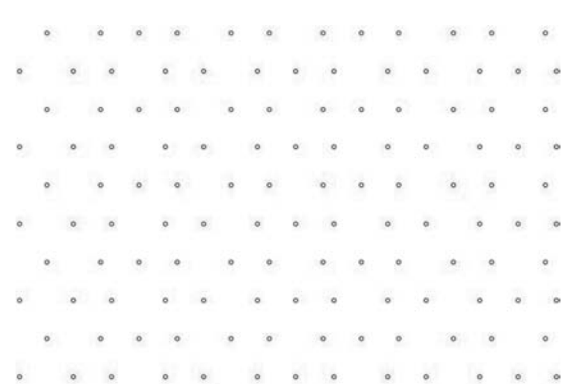

Figure 2
Part of the basal plane (xy) quasiperiodic in the horizontal x direction and periodic in the vertical y direction.

The product is a three-dimensional crystal structure periodic in one direction (say z) and quasiperiodic in the basal planes (xy) perpendicular to it. It is a layer structure formed by periodic repetition of six plane layers denoted by $A_0A_1A_2A_3\bar{A}_2\bar{A}_1$. Layers of the types A_0 and A_3 show sixfold symmetry (Figs 3. and 4) while the rest are only threefold symmetric (Figs. 5 and 6). Layers $\bar{A}_1$ and $\bar{A}_2$ are the inversions of A_1 and A_2, respectively. The whole structure has point symmetry $\bar{3}m$ around the origin.

Collapsing the layers onto the basal plane (xy) produces the final two-dimensional projection (Fig. 7). It has only sixfold symmetry. However, it does contain twelvefold subsets of points and, moreover, the dominant motif is a dodecagon.

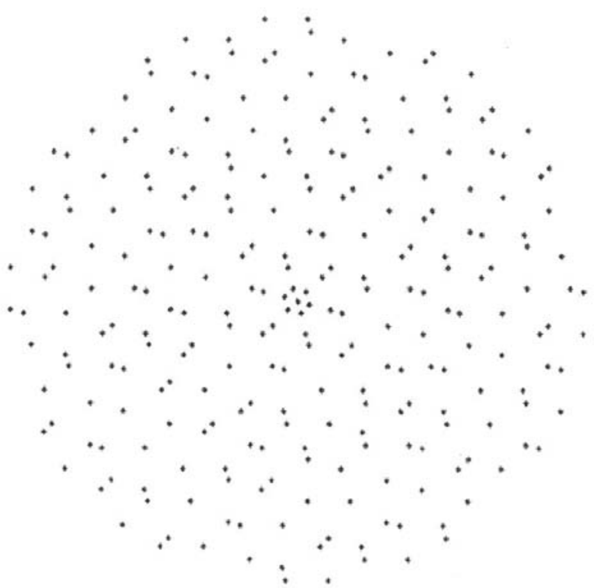

Figure 3
Sixfold symmetric layer A_0.

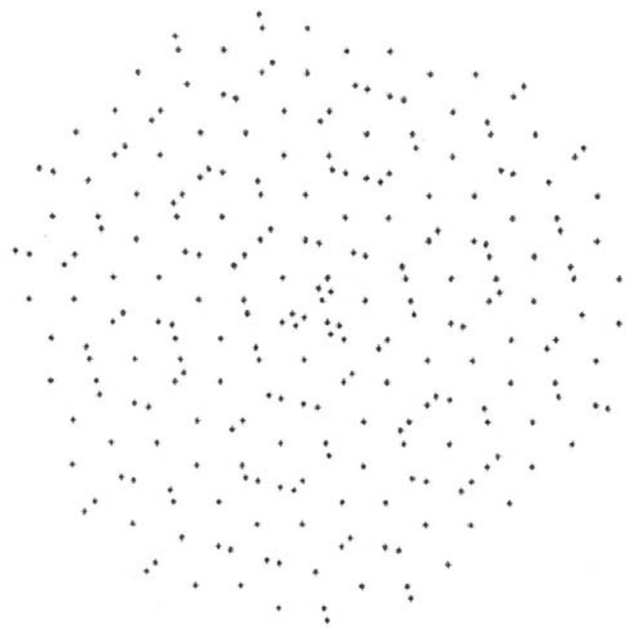

Figure 5
Threefold symmetric layer A_1.

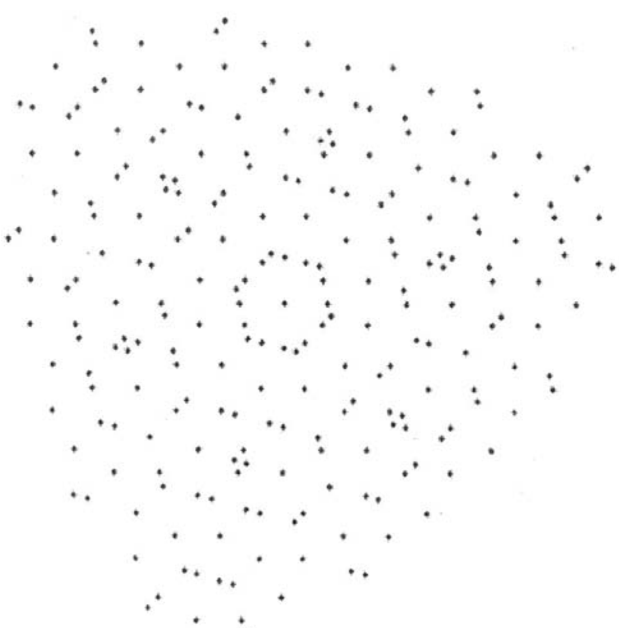

Figure 4
Sixfold symmetric layer A_3.

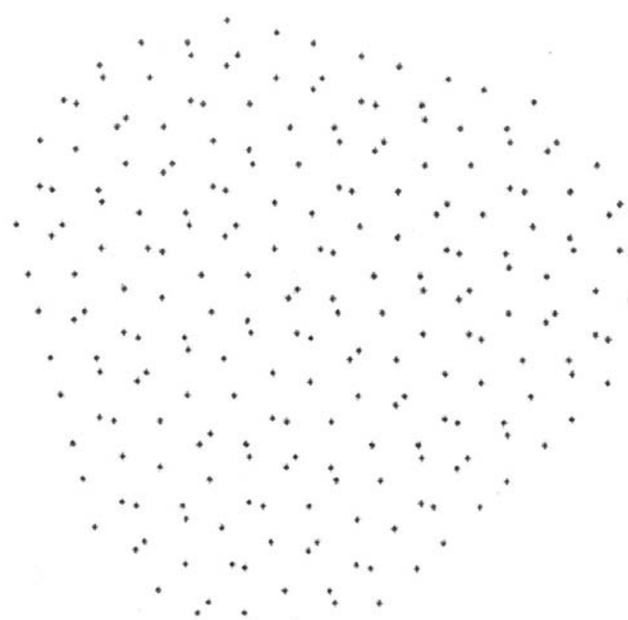

Figure 6
Threefold symmetric layer A_2.

5. Dodecagonal structure

We start with the root lattice D_4, alias the four-dimensional checkerboard lattice, better known as the four-dimensional (both body- and face-) centered cubic lattice (Baake *et al.*, 1991; Conway & Sloane, 1999). This lattice is interesting by itself, being the densest and most symmetric of all four-dimensional lattices. Its symmetries include the rotations 1, 2, 3, 4, 6, 8 and 12. Every lattice point has 24 nearest neighbors. The latter form a 24-cell, alias a polytope {3,4,3} (*cf.* Coxeter, 1973). Since this is self-dual, the Voronoi domain (also known as the Wigner–Seitz cell or symmetric unit cell) around a lattice point is a 24-cell as well. However, in the same three-dimensional projection, the images of these two 24-cells turn out to be different. We consider a central domain $\boldsymbol{C}$ formed by the origin and its 24 nearest neighbors. The symmetry of $\boldsymbol{C}_3$, the three-dimensional projection of the central domain $\boldsymbol{C}$ is *mmm*. The projection of D_4 onto three dimensions is constrained by the requirement that the second two-dimensional projection of $\boldsymbol{C}$ be a regular dodecagon centered around the origin (Coxeter, 1973). Thus, as a result, the three-dimensional projection of D_4 turns out to be a hybrid structure that is quasiperiodic in one direction and periodic in the plane perpendicular to it. The second cut and projection then yields a two-dimensional dodecagonal pattern.

The root lattice D_4 is defined by

$$D_4 := \left\{[x^1, x^2, x^3, x^4] \in \mathbf{Z}^4 : x^1 + x^2 + x^3 + x^4 \in 2\mathbf{Z}\right\}. \tag{7}$$

In other words, the lattice points are integral multiples of $\langle 1100 \rangle$.

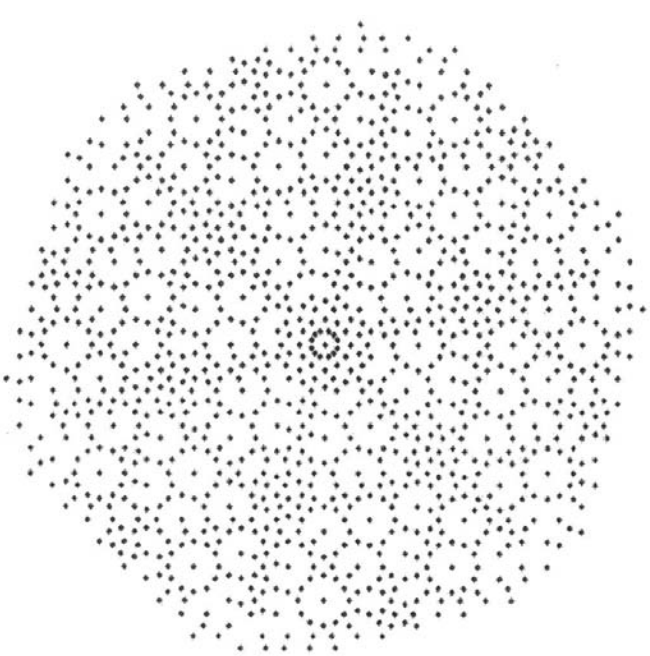

Figure 7
Projection of all layers onto the basal plane.

We choose the four-dimensional orthonormal basis as

$$\begin{aligned} \mathbf{R}_{\|}^3 &: \begin{cases} \mathbf{e}_1 = \left[\dfrac{a}{2}, \dfrac{a}{2}, 0, \dfrac{b}{2^{1/2}}\right], \\ \mathbf{e}_2 = \left[\dfrac{a}{2}, -\dfrac{a}{2}, \dfrac{b}{2^{1/2}}, 0\right], \\ \mathbf{e}_3 = \left[\dfrac{b}{2^{1/2}}, 0, -\dfrac{a}{2}, -\dfrac{a}{2}\right], \end{cases} \\ \mathbf{R}_{\perp}^1 &: \quad \mathbf{e}_4 = \left[0, \dfrac{b}{2^{1/2}}, \dfrac{a}{2}, -\dfrac{a}{2}\right], \end{aligned} \tag{8}$$

where

$$\begin{aligned} & a^2 = 1 + \frac{1}{3^{1/2}}, \quad b^2 = 1 - \frac{1}{3^{1/2}}, \\ & \frac{a}{b} = \frac{\cos(\pi/3)}{\cos(5\pi/12)} = (2 + 3^{1/2})^{1/2} = 1.931\,851\,6\ldots, \end{aligned} \tag{9}$$

corresponding to the twelvefold symmetry.

The window is either the half-open interval

$$\mathbf{W} := \left(-\left(\frac{a}{2} + \frac{b}{2^{1/2}}\right), +\left(\frac{a}{2} + \frac{b}{2^{1/2}}\right)\right] \subset \mathbf{R}_{\perp}^1 \tag{10}$$

or

$$\mathbf{W}' := \left[-\left(\frac{a}{2} + \frac{b}{2^{1/2}}\right), +\left(\frac{a}{2} + \frac{b}{2^{1/2}}\right)\right) \subset \mathbf{R}_{\perp}^1. \tag{10$'$}$$

The projection into three dimensions results in a layer structure that is best represented in transformed coordinates:

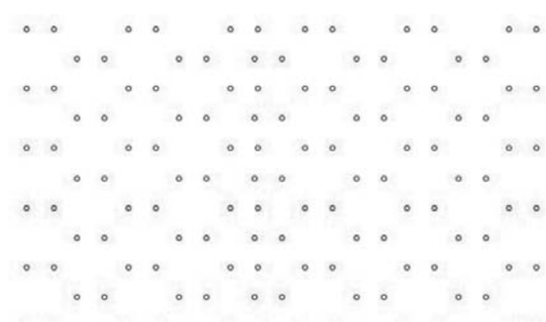

Figure 8
Equatorial layer ($z = 0$).

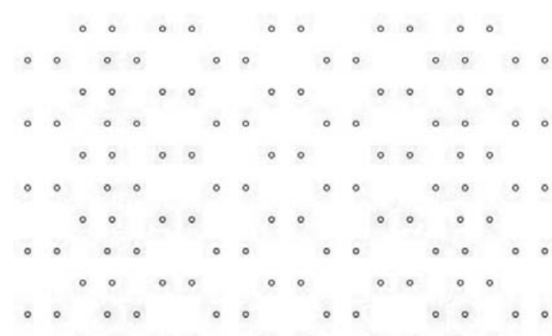

Figure 9
Top or bottom layer ($z = \pm 1$).

$$
\begin{cases}
\mathbf{R}^3_{||}: & \left\{\begin{aligned}
\mathbf{e}'_1 &= \left[\frac{b}{2^{1/2}}, 0, 0, \frac{a}{2^{1/2}}\right], \\
\mathbf{e}'_2 &= \left[0, \left(\frac{a^2+b^2}{2}\right)^{1/2}, 0, 0\right], \\
\mathbf{e}'_3 &= \left[0, 0, \left(\frac{a^2+b^2}{2}\right)^{1/2}, 0\right],
\end{aligned}\right. \\
\mathbf{R}^1_{\perp}: & \quad \mathbf{e}'_4 = \left[-\frac{a}{2^{1/2}}, 0, 0, \frac{b}{2^{1/2}}\right].
\end{cases} \tag{11}
$$

The structure is quasiperiodic in one direction, x, along $\mathbf{e}'_1$, with quasiperiods $b = (1 - 1/3^{1/2})^{1/2}$ and $3^{1/2}b = (3 - 3^{1/2})^{1/2}$. It is periodic in the perpendicular plane (yz) spanned by $\mathbf{e}'_2$ and $\mathbf{e}'_3$ with y period $2^{1/2}$ and z period 2. Strictly speaking, two kinds of layers alternate: one contains the equatorial section of $\boldsymbol{C}_3$, the three-dimensional projection of the central unit cell $\boldsymbol{C}$, the other the top (or bottom) face (Figs. 8 and 9). They are, of course, indistinguishable (*cf.* Baake, 1999).

A second cut and orthogonal projection along the direction of $\mathbf{e}_3$ produces the final two-dimensional dodecagonal structure. Fig. 10 shows the projection of the first three shells around the origin of D_4 upon the basal plane. The points are labeled by their four-dimensional Cartesian coordinates. The four-dimensional nearest-neighbor shell is a 24-cell. Its vertices are numbered ***01*** through ***24***. Fig. 11 shows how the skew layers project onto the final twelvefold basal plane.

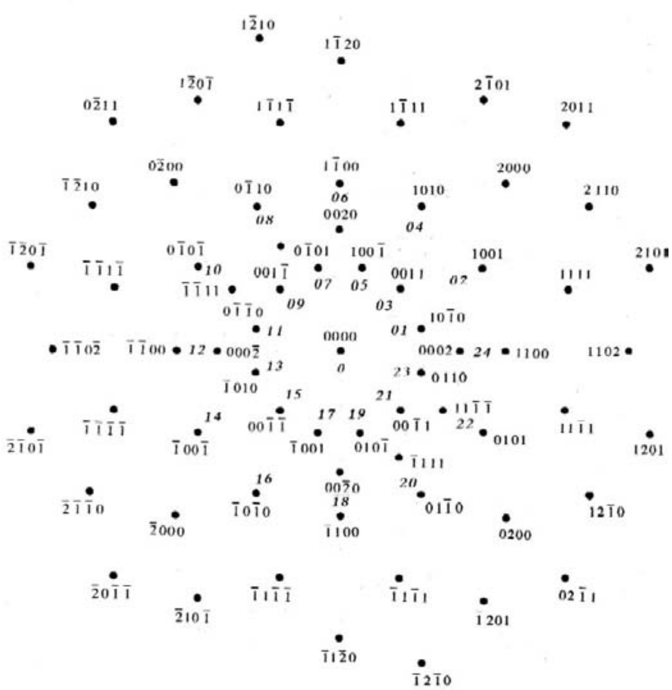

Figure 10
Final two-dimensional twelvefold structure. Projection of the first three shells. Points are labeled by their four-dimensional coordinates.

6. 'Straightforward' pentagonal/decagonal structure

In the original cut-and-project approach to the Penrose tiling and its generalizations, the five-dimensional simple cubic lattice $\mathbf{Z}^5$ was projected onto a suitable plane $\mathbf{R}^2$ as parallel (physical) space while the window was in its complementary three-space $\mathbf{R}^3$ [*cf.* Gähler & Rhyner (1986) or Senechal (1995)]. Here we reverse the roles. We focus on the structure in the parallel three-space $\mathbf{R}^3_{||}$. The convex hull of $\boldsymbol{C}_3$, the three-dimensional projection of the five-dimensional central unit cell $\boldsymbol{C}$, is a rhombic icosahedron; apart from that, $\boldsymbol{C}_3$ contains ten inner points. Although the projection from five

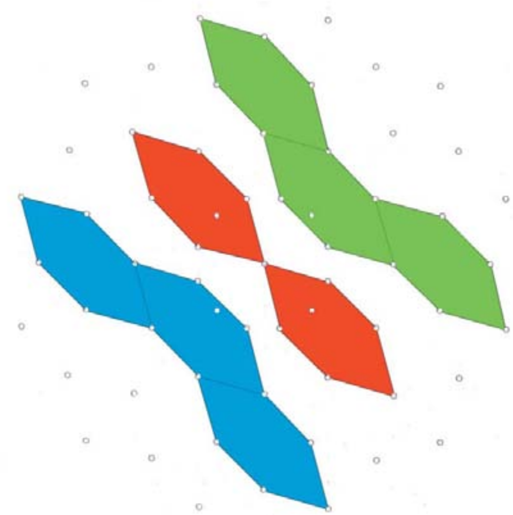

Figure 11
Skew layers projected onto the twelvefold basal plane.

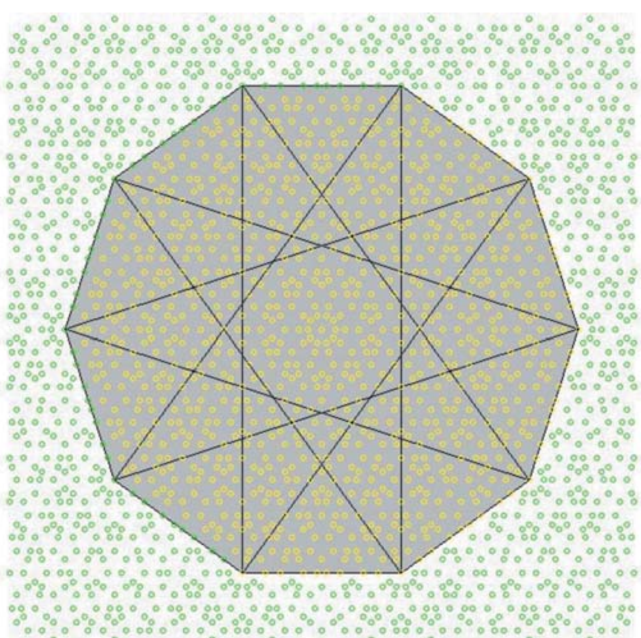

Figure 12
Window for 'straightforward' decagonal structure. The yellow points refer to lattice points, whose projections are located inside the acceptance domain (grey), whereas the green points indicate discarded lattice points.

dimensions is not minimal (Baake *et al.*, 1990), it appears to be the most convenient and transparent one.

The five-dimensional simple cubic lattice will be represented by $(2\mathbf{Z}+1)^5$ Hence, the central unit cell $\boldsymbol{C}$ is spanned by the vectors $\langle 11111\rangle$.

We choose as our orthonormal base the vectors

$$
\begin{aligned}
&\mathbf{R}_0^1: \quad \mathbf{e}_0 = \frac{1}{5^{1/2}}[11111],\\
&\mathbf{R}_{||}^2: \begin{cases} \mathbf{e}_1 = \left(\dfrac{2}{5}\right)^{1/2}[1cCCc], \\ \mathbf{e}_2 = \left(\dfrac{2}{5}\right)^{1/2}\left[0sS\bar{S}\bar{s}\right], \end{cases}\\
&\mathbf{R}_{\perp}^2: \begin{cases} \mathbf{e}_3 = \left(\dfrac{2}{5}\right)^{1/2}[1CccC], \\ \mathbf{e}_4 = \left(\dfrac{2}{5}\right)^{1/2}\left[0S\bar{s}s\bar{S}\right], \end{cases}
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
c &:= \cos\frac{2\pi}{5} = \frac{\tau^{-1}}{2}, \quad s := \sin\frac{2\pi}{5} = \frac{(2+\tau)1/2}{2},\\
C &:= \cos\frac{4\pi}{5} = -\frac{\tau}{2}, \quad S := \sin\frac{4\pi}{5} = \frac{(3-\tau)^{1/2}}{2},\\
\bar{c} &:= -c, \quad \bar{C} := -C, \quad \bar{s} := -s, \quad \bar{S} := -S,
\end{aligned}
\tag{13}
$$

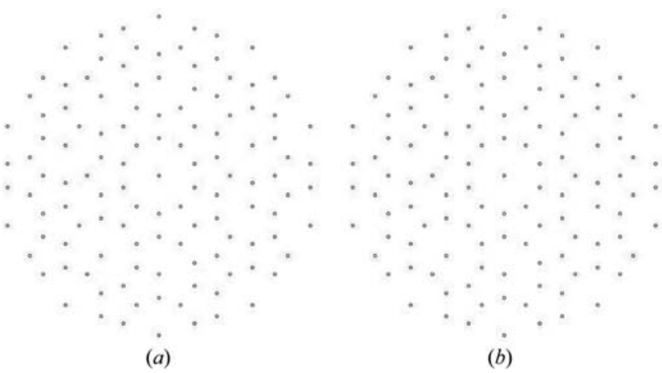

Figure 13
Tenfold layers $A_5\,(z = +5^{1/2})$ and $\bar{A}_5\,(z = -5^{1/2})$.

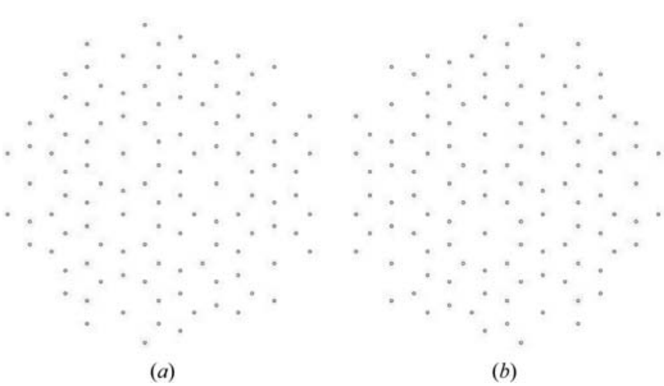

Figure 14
Fivefold layers $A_3\,(z = +3/5^{1/2})$ and $\bar{A}_3\,(z = -3/5^{1/2})$.

and, as usual,

$$\tau := \frac{1+5^{1/2}}{2} \tag{14}$$

is the golden mean.

In this representation, the standard base becomes

$$
\begin{aligned}
[10000] &\rightarrow \left(\frac{2}{5}\right)^{1/2}\left[\frac{1}{2^{1/2}}; 1, 0; 1, 0\right],\\
[01000] &\rightarrow \left(\frac{2}{5}\right)^{1/2}\left[\frac{1}{2^{1/2}}; c, s; C, S\right],\\
[00100] &\rightarrow \left(\frac{2}{5}\right)^{1/2}\left[\frac{1}{2^{1/2}}; C, S; c, \bar{s}\right],\\
[00010] &\rightarrow \left(\frac{2}{5}\right)^{1/2}\left[\frac{1}{2^{1/2}}; C, \bar{S}; c, s\right],\\
[00001] &\rightarrow \left(\frac{2}{5}\right)^{1/2}\left[\frac{1}{2^{1/2}}; c, \bar{s}; C, \bar{S}\right].
\end{aligned}
\tag{15}
$$

The window $\boldsymbol{W}$ is the two-dimensional projection of the central unit cell $\boldsymbol{C}$ onto the perpendicular space $\mathbf{R}_{\perp}^2$; it is, of course, a decagon (Fig. 12).

The cut-and-project procedure into the three-dimensional space $\mathbf{R}_{||}^3 = \mathbf{R}_0^1 \times \mathbf{R}_{||}^2$ produces a layer structure. It is quasiperiodic in the (***12***) plane, alias $\mathbf{R}_{||}^2$, and periodic in its perpendicular direction ***0***, alias $\mathbf{R}_0^1$. The respective quasiperiods are 1 and τ, the period is $2 \times 5^{1/2}$. The layer spacing is $2/5^{1/2}$. The point symmetry, strictly speaking, around the origin is $\bar{5}m$. It is, however, slightly broken. Certain points are absent since their projections into the perpendicular space fall outside the window. Their projections into parallel space $\mathbf{R}_{||}^2$ would be too close to some other point. This phenomenon is well known in quasiperiodic structures. At any rate, of course, the symmetry must be understood in a generalized sense, namely that of indistinguishability.

We wish to point out that this structure has been put forward earlier in a different context by Lück (1987).

A second cut and orthogonal projection in the direction ***0*** produces, as expected, a two-dimensional quasiperiodic decagonal structure. This structure is, of course, the product of collapsing all layers onto a single plane. Figs. 13 through 16 show central patches of the layers $\bar{A}_5\,(z = -5^{1/2})$ through

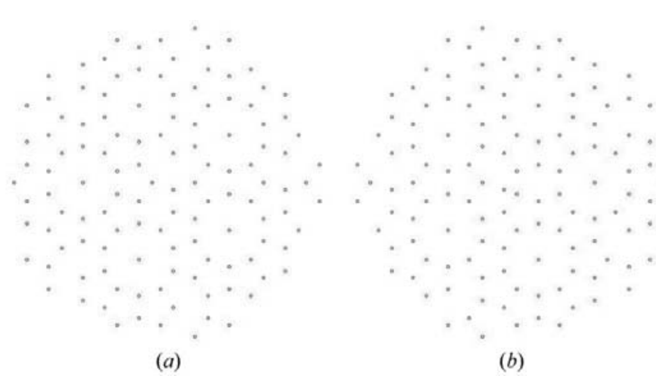

Figure 15
Fivefold layers $A_1\,(z = +1/5^{1/2})$ and $\bar{A}_1\,(z = -1/5^{1/2})$.

$A_5\,(z = +5^{1/2})$ as well as all layers collapsed onto a single plane. Again, layer $\bar{A}_i$ is the inversion of layer A_i (up to indistinguishability). Layers A_1, $\bar{A}_1$, A_3 and $\bar{A}_3$ are fivefold, while layer A_5 is tenfold being its own inversion: $A_5 \equiv \bar{A}_5$.

7. 'Minimal' pentagonal/decagonal structure – monoclinic variant

To construct a fivefold or tenfold pattern, it is not necessary to start from five dimensions; four suffice (Baake *et al.*, 1990). Here we present the 'minimal' three-dimensional structure with fivefold (geneneralized) point symmetry which comes in two variants.

We start with the root lattice A_4, conventionally and conveniently represented using five coordinates:

$$A_4 := \left\{[x^0, x^1, x^2, x^3, x^4] \in \mathbf{Z}^5 : x^0 + x^1 + x^2 + x^3 + x^4 = 0\right\}. \qquad (16)$$

The five-dimensional orthonormal base and the standard unit vectors are again given by equations (11) through (14). We consider a four-dimensional central domain $\boldsymbol{C}$ (rather than a unit cell) spanned by the vectors $\langle 1\bar{1}000 \rangle$. It includes the origin and its 20 nearest neighbors.

The lattice lies entirely in the four-dimensional space $\mathbf{R}^4 = \mathbf{R}^2_{||} \times \mathbf{R}^2_{\perp}$ orthogonal to $\mathbf{R}^1_0$. This time we choose $\mathbf{R}^3_{||} = (\boldsymbol{123})$ spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as parallel space and the axis $\boldsymbol{4}$ along $\mathbf{e}_4$ as perpendicular space $\mathbf{R}^1_{\perp}$.

The window is either the half-open interval

$$\boldsymbol{W} := \left(-\left(\frac{8}{5}\right)^{1/2} s, +\left(\frac{8}{5}\right)^{1/2} s\right] \subset \mathbf{R}^1_{\perp} \qquad (17)$$

or

$$\boldsymbol{W}' := \left[-\left(\frac{8}{5}\right)^{1/2} s, +\left(\frac{8}{5}\right)^{1/2} s\right) \subset \mathbf{R}^1_{\perp}. \qquad (17')$$

The cut and projection into three dimensions produces a monoclinic hybrid structure. The direction $\boldsymbol{2}$ turns out to be quasiperiodic; it will hence be renamed x. In the perpendicular $(\boldsymbol{13})$ plane, now renamed (yz), the structure is periodic. It is oblique with an angle $\beta = \arccos(1/3^{1/2}) = 65^\circ\,54'\,18.57\ldots''$. There are two kinds of layers: an equatorial layer at $z = 0$ and

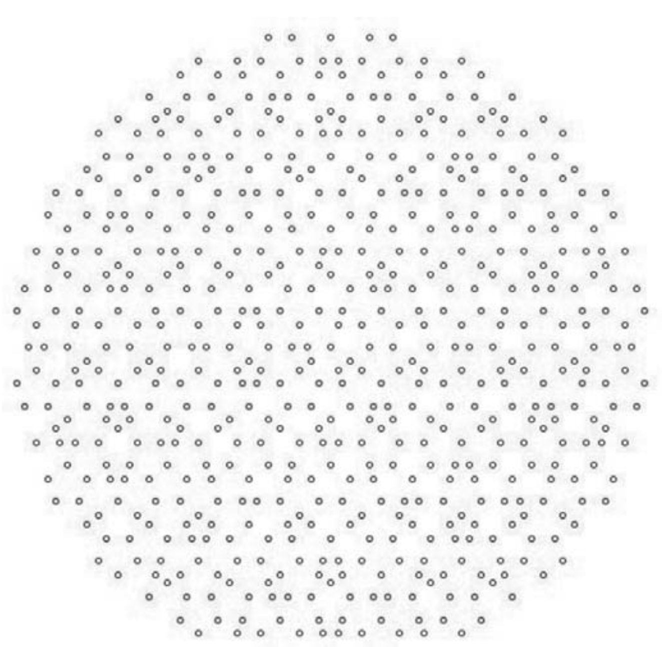

Figure 16
The two-dimensional structure – all layers collapsed.

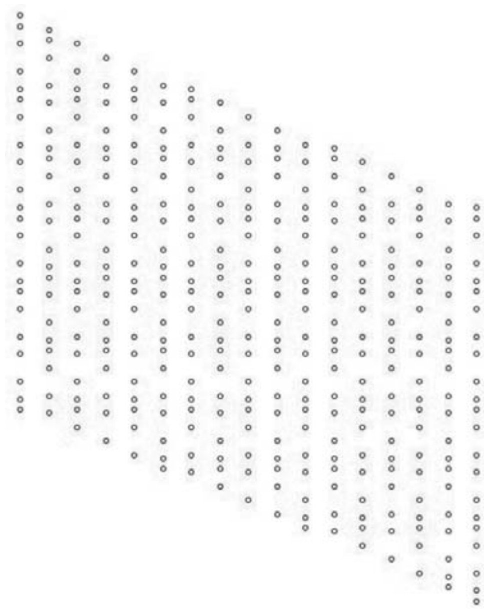

Figure 17
Equatorial layer ($z = 0$).

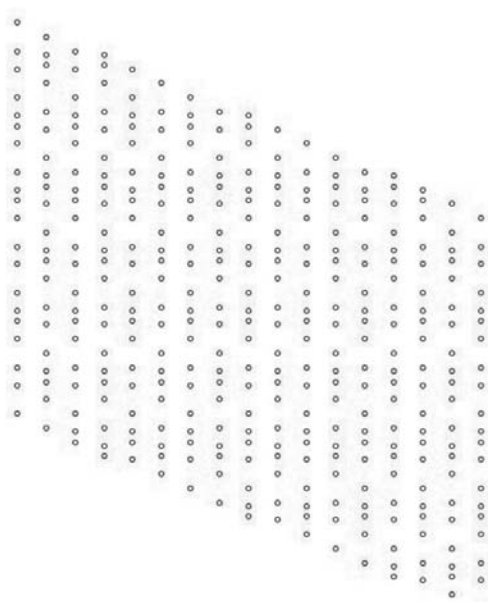

Figure 18
Top or bottom layer [$z = \pm(5/6)^{1/2}$].

a top (or bottom) layer at $z = (5/6)^{1/2}$, which are physically indistinguishable. Those layers are shown in Figs. 17 and 18. Note that some points at the border of these oblique patches are missing due to limited computational resources. But the periods are clearly observable, which are best represented by the repeat vectors. These are $[2\bar{1}00\bar{1}]$ and $[20\bar{1}\bar{1}0]$; both are in the (yz) plane and of three-dimensional length $6^{1/2}$. The quasiperiods are $[001\bar{1}0]$, of three-dimensional length $(8/5)^{1/2}s$, and $[0100\bar{1}]$, of three-dimensional length $(8/5)^{1/2}S$; their ratio is $s/S = \tau$, as expected. Thus, the points along x form Fibonacci chains.

The second cut and projection produces a two-dimensional point set possessing fivefold symmetry (Baake *et al.*, 1990). It is worth noting that the structure is associated with the number field $\mathbf{Q}(2^{1/2}, 3^{1/2}, 5^{1/2})$, that is, with the quadratic irrationals characteristic of all known quasicrystals: pentagonal and decagonal, including icosahedral, as well as octagonal and dodecagonal.

8. 'Minimal' pentagonal/decagonal structure – tetragonal variant

In this instance, we choose $\mathbf{R}^3_{||} = (\boldsymbol{124})$ spanned by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4$ as parallel space and the axis $\boldsymbol{3}$ along $\mathbf{e}_3$ as perpendicular space $\mathbf{R}^1_\perp$.

The window is either the half-open interval

$$\boldsymbol{W} := \left(\left(\frac{2}{5}\right)^{1/2}(C-c), \left(\frac{2}{5}\right)^{1/2}(c-C)\right] = \left(-\frac{1}{2^{1/2}}, +\frac{1}{2^{1/2}}\right] \subset \mathbf{R}^1_\perp \tag{18}$$

or

$$\boldsymbol{W}' := \left[\left(\frac{2}{5}\right)^{1/2}(C-c), \left(\frac{2}{5}\right)^{1/2}(c-C)\right) = \left[-\frac{1}{2^{1/2}}, +\frac{1}{2^{1/2}}\right) \subset \mathbf{R}^1_\perp. \tag{18'}$$

The cut and projection into three dimensions produces a tetragonal hybrid structure. It is in some sense dual to the monoclinic variant; namely, the periods and quasiperiods are interchanged. Now the direction $\boldsymbol{1}$ turns out to be quasiperiodic; it will hence be renamed x. In the perpendicular $(\boldsymbol{24})$ plane, now renamed (yz), the structure is periodic. The periods are $[001\bar{1}0]$ and $[0100\bar{1}]$, both of three-dimensional length $2^{1/2}$. They are mutually orthogonal both in four dimensions and in three dimensions. The quasiperiods are $[2\bar{1}00\bar{1}]$, of three-dimensional length $(8/5)^{1/2}(1-c)$, and $[20\bar{1}\bar{1}0]$, of three-dimensional length $(8/5)^{1/2}(1-C)$; their ratio is $(1-C)/(1-c) = \tau^2 = \tau + 1$. Thus, the points along x again form Fibonacci chains. There are again two indistinguishable layers at $z = 0$ and $z = 1/2^{1/2}$, as shown in Figs. 19 and 20 (same remarks as for Figs. 17 and 18). The second cut and projection produces, of course, the same two-dimensional point set as that derived from the monoclinic variant.

9. Conclusions

We have put forward a variant of the cut-and-project method by performing it in two stages, starting from a periodic structure in a space of higher dimension ($D > 3$) through three-dimensional space eventually to arrive at a plane quasiperiodic structure. Our purpose was to construct structures in which the periodic dimensions are intrinsically connected with the quasiperiodic ones. We have applied the method to the known instances of 'two-dimensional' quasicrystals: pentagonal/decagonal, octagonal and dodecagonal. Our results may be useful for the interpretation of the latter as well as of modulated structures. The octagonal case, while hardly realistic, may serve well as a simple model for phasons. Apart from that, the method has some intrinsic merits for the mathematics of quasiperiodic structures. It can be directly applied to other symmetries. The starting and the intermediate dimension may be easily changed and the number of stages may be increased.

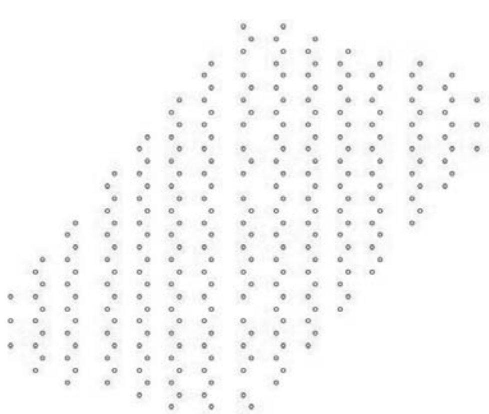

Figure 19
Equatorial layer ($z = 0$).

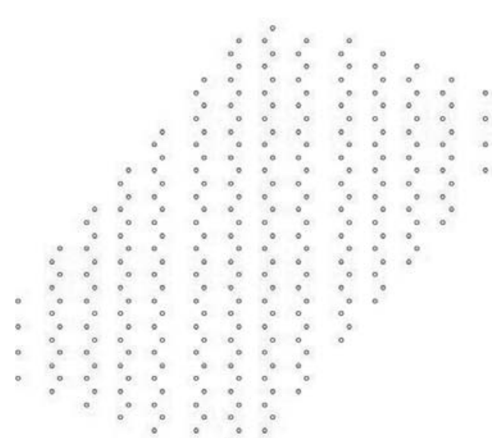

Figure 20
Top or bottom layer ($z = \pm 1/2^{1/2}$).

References

Baake, M. (1999). *A Guide to Mathematical Quasicrystals*. *Quasicrystals – an Introduction to Structure, Physical Properties and Applications*, edited by J.-B. Suck, M. Schreiber & P. Häussler, pp. 17–48. Berlin: Springer (2002), http://ArXiv.org/abs/math-ph/9901014.

Baake, M., Joseph, D. & Schlottmann, M. (1991). *Int. J. Mod. Phys.* B**5**, 1927–1953.

Baake, M., Kramer, P., Schlottmann, M. & Zeidler, D. (1990). *Int. J. Mod. Phys.* B**4**, 2217–2268.

Ben-Abraham, S. I. (2004). *Ferroelectrics*, **305**, 29–32.

Ben-Abraham, S. I., Lerer, Y. & Sapir, Y. (2004). *J. Non-Cryst. Solids*, **334&335**, 71–76.

Conway, J. H. & Sloane, N. J. A. (1999). *Sphere Packings, Lattices and Groups*, 3rd ed. New York: Springer.

Coxeter, H. S. M. (1973). *Regular Polytopes*, 3rd ed. New York: Dover.

Gähler, F. & Rhyner, J. (1986). *J. Phys. A: Math. Gen.* **19**, 267–277.

Janot, C. (1994). *Quasicrystals: A Primer*, 2nd ed. Oxford: Clarendon Press.

Lück, R. (1987). *Mater. Sci. Forum*, **22–24**, 231–246.

Moody, R. V. (1997). *Meyer Sets and Their Duals*. *The Mathematics of Long-Range Aperiodic Order*, edited by R. V. Moody, pp. 403–441. Dordrecht: Kluwer.

Senechal, M. (1995). *Quasicrystals and Geometry*. Cambridge University Press.